Characterization of hyperbolic Landau states by coherent state transforms

2003 J. Phys. A: Math. Gen. 36 8071

(http://iopscience.iop.org/0305-4470/36/29/311)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 8071–8076 PII: S0305-4470(03)57851-9

Characterization of hyperbolic Landau states by coherent state transforms

Z Mouayn

Department of Mathematics, Faculty of Sciences and Technics (M'Ghila),
Cadi Ayyad University, BP 523, Béni Mellal, Morocco

E-mail: mouayn@math.net

Received 3 January 2003, in final form 16 May 2003
Published 8 July 2003
Online at stacks.iop.org/JPhysA/36/8071

Abstract

We deal with a family of generalized coherent states obtained by means of operators of an unitary irreducible representation of the group of affine transformations of the real line. We prove that the ranges of the corresponding coherent state transforms coincide with spaces of bound states of the Landau Hamiltonian in the hyperbolic plane. This provides us with a new characterization of hyperbolic Landau states.

PACS numbers: 02.20.−a, 03.65.Fd

1. Introduction

In quantum mechanics systems, the canonical commutation relation $[X, Y] = -\mathrm{i}I$ plays an important role, and the annihilation and creation operators, coherent states and number states are derived from this commutation relation [1]. Another important commutation relation is $[X, Y] = -\mathrm{i}X$. The Lie group associated with this algebra is the semi-direct product $G := \mathbf{R}_+^* \propto \mathbf{R}$, called the affine group. The corresponding 'affine' coherent states (ACS) are well known [2]. Wave packets of these ACS are important in the theory of the continuous wavelet transformation [3].

In this paper, we are concerned with an infinite-dimensional unitary irreducible representation (UIR) of G realized on the Hilbert space $\mathcal{H}$ of square integrable functions on the group $\mathbf{R}_+^*$ endowed with its Haar measure $\lambda^{-1}\,\mathrm{d}\lambda$ [4]. We displace via operators of this representation a special vector of $\mathcal{H}$ (chosen as vacuum state) to obtain a family of generalized affine coherent states (see [5] for the group theoretical formalism). Identifying the group G with the complex upper-half $\mathbf{H}^2$, we establish that the image of the Hilbert space $\mathcal{H}$ under the coherent state transforms associated with the constructed coherent states coincides with spaces of bound states of the Landau Hamiltonian on $\mathbf{H}^2$ [6].

This paper is organized as follows. In section 2, we review the definition of the affine group as well as the square integrabilty property of the representation we are dealing with. Section 3 deals with generalized affine coherent states. In section 4 we establish a characterization theorem for spaces of hyperbolic Landau states by means of the corresponding coherent state transforms. Section 5 is devoted to some concluding remarks.

2. The affine group

The affine group $G := \mathbf{R} \times \mathbf{R}_+^*$ consists of elements $(x, y), x$ real and $y > 0$ acting on elements t of the real line according to: $t \to yt + x$. The group law of G is: $(x, y) \cdot (x', y') = (x + yx', yy')$. G is a locally compact nonunimodular group with the left Haar measure $\mathrm{d}\mu(x, y) = (2\pi y)^{-2}\,\mathrm{d}x\,\mathrm{d}y$ and modular function $\Delta(x, y) = y^{-1}$.

This group has two inequivalent infinite dimensional unitary irreducible representations UIR (see [7] for the complete description of the dual $\widehat{G}$ of G). We shall consider one of these UIR, denoted π_+, realized on the Hilbert space $\mathcal{H} = L^2(\mathbf{R}_+^*, \lambda^{-1}\,\mathrm{d}\lambda)$ as:

$$\pi_+(x, y)[\varphi](\lambda) := \exp\left(\tfrac{1}{2}\mathrm{i}\lambda x\right)\varphi(\lambda y) \qquad \varphi \in \mathcal{H} \qquad \lambda \in \mathbf{R}_+^*. \tag{2.1}$$

This representation is square integrable. i.e., there exists a vector $\phi_0 \in \mathcal{H}$ such that the function: $(x, y) \to \langle \pi_+(x, y)\phi_0, \phi_0\rangle_{\mathcal{H}}$ belongs to $L^2(G, \mathrm{d}\mu)$. This condition can also be expressed through the existence of an operator K in $\mathcal{H}$ self-adjoint, positive and semi-invariant with weight Δ^{-1} such that

$$\int_G \mathrm{d}\mu(x, y)\langle \varphi_1, \pi_+(x, y)\psi_1\rangle\langle \pi_+(x, y)\psi_2, \varphi_2\rangle = \langle \varphi_1, \varphi_2\rangle\left\langle K^{\frac{1}{2}}\psi_1, K^{\frac{1}{2}}\psi_2\right\rangle \tag{2.2}$$

for all $\psi_1, \psi_2 \in \mathrm{dom}\left(K^{\frac{1}{2}}\right)$ and all $\varphi_1, \varphi_2 \in \mathcal{H}$. Precisely, $K^{\frac{1}{2}}[\psi](\lambda) = \lambda^{-\frac{1}{2}}\psi(\lambda), \psi \in \mathrm{dom}\left(K^{\frac{1}{2}}\right)$, and we note that the unboundedness of the operator K is due to the fact that the group G is not unimodular ([8], p 215).

3. Generalized affine coherent states

According to ([5], p 56), a Hilbert space H has a system $\{f_g\}$ of coherent states, labelled by elements g of a group G if:

(i) there is a representation $T : g \to T(g)$ of G by unitary operators $T(g)$ on H,
(ii) there is a vector $f_0 \in H$ such that for $f_g = T_g f_0$ and arbitrary $f \in H$ we have

$$\langle f, f\rangle_H = \int_G \mathrm{d}\nu(g)|\langle f, f_g\rangle|^2 \tag{3.1}$$

$\mathrm{d}\nu$ being the left Haar measure of G. Note that a polarization of (3.1) gives us the equality

$$\langle f_1, f_2\rangle_H = \int_G \mathrm{d}\nu(g)\langle f_1, f_g\rangle\overline{\langle f_2, f_g\rangle}. \tag{3.2}$$

Now, according to the above definition, we consider a set of coherent states, denoted $|(x, y)\rangle_{b,m}, b > 0$, obtained by displacing via the representation operator $\pi_+(x, y)$ the vector $|0\rangle_{b,m}$ of $\mathcal{H}$ having the following wavefunction

$$\langle \lambda \mid 0\rangle_{b,m} := \left(\frac{\Gamma(2b - m)}{m!}\right)^{-\frac{1}{2}} \lambda^{b-m}\,\mathrm{e}^{-\frac{1}{2}\lambda} L_m^{2(b-m)-1}(\lambda)$$

where $L_n^{\alpha}(.)$ is the Laguerre polynomial ([9], p 1037). Precisely

$$|(x, y)\rangle_{b,m} := \pi(x, y)|0\rangle_{b,m}$$

and wavefunctions are given by

$$\langle\lambda \mid (x, y)\rangle_{b,m} = \left(\frac{\Gamma(2b-m)}{m!}\right)^{-\frac{1}{2}} (\lambda y)^{b-m} \, \mathrm{e}^{-\frac{1}{2}\lambda y} \exp\left(\frac{1}{2}\mathrm{i}\lambda x\right) L_m^{2(b-m)-1}(\lambda y).$$

These coherent states are, up to a multiplicative constant, completely justified by the square integrability of the UIR π_+ since condition (3.2) in the above definition follows from equality (2.2). The latter implies that the unity of $\mathcal{H}$ is solved as

$$\mathbf{1}_{\mathcal{H}} = c_{b,m}^{-1} \int_G \mathrm{d}\mu(x, y) |(x, y)\rangle_{b,m} \;\; {}_{b,m}\langle (x, y)|$$

where the constant $c_{b,m}$ equals the norm square in $\mathcal{H}$ of the state $K^{\frac{1}{2}}|0\rangle_{b,m}$.

Note that for $m = 0$, the state $|(x, y)\rangle_{b,0}$ coincides with the well-known affine coherent state [2]. For $m \neq 0$, we should note that the constructed states $|(x, y)\rangle_{b,m}$ are identical to a continuous wavelet family discussed in [10].

4. A characterization of Landau hyperbolic states

Here, we identify G with the complex upper half-plane $\mathbf{H}^2 = \{\xi \in \mathbf{C}, \operatorname{Im}\xi > 0\}$ by setting $\xi = x + \mathrm{i}y \equiv (x, y)$. So that the space $L^2(G, \mathrm{d}\mu)$ coincides with the space $L^2(\mathbf{H}^2)$ endowed with the norm $\|\phi\|^2 = \int_{\mathbf{H}^2} |\phi(\xi)|^2 (\operatorname{Im}\xi)^{-2} \, \mathrm{d}\operatorname{Re}\xi \, \mathrm{d}\operatorname{Im}\xi$.

We consider the coherent state transform $\mathcal{W}_{b,m} : \mathcal{H} \to L^2(\mathbf{H}^2)$ defined in the usual way by

$$\mathcal{W}_{b,m}[\varphi](\xi) := c_{b,m}^{-\frac{1}{2}} \int_{\mathbf{R}_+^*} \frac{\mathrm{d}\lambda}{\lambda} (\varphi(\lambda))^* \langle\lambda \mid \xi\rangle_{b,m} \qquad \varphi \in \mathcal{H} \qquad \xi \in \mathbf{H}^2.$$

Thanks to the square integrability of the UIR π_+, $\mathcal{W}_{b,m}$ is an isometrical embedding. But to determine the range of this isometry, we need first to fix some notation. For a real number B and $m \in \mathbf{Z}_+$ with $0 \leqslant m < |B| - 1/2$, we denote by $E_{B,m}$ the subspace of $L^2(\mathbf{H}^2)$ defined by

$$E_{B,m} := \{\psi \in L^2(\mathbf{H}^2, \mathrm{d}\mu), H_B\psi = e_m\psi\}$$

where $e_m := (|B| - m)(|B| - m - 1)$ and

$$H_B := y^2\left(\partial_x^2 + \partial_y^2\right) - 2\mathrm{i}By\partial_x \tag{4.1}$$

In suitable units, the operator in (4.1) represents the Landau Hamiltonian of a charged particle in a uniform magnetic field on $\mathbf{H}^2$ with magnetic length proportional to $|B|$. The operator H_B, also called Maass Laplacian, has been discussed in the mathematical and physical literature (see [11–14]). The defined spaces $E_{B,m}$ are eigenspaces of H_B in $\mathcal{H}$ associated with Landau hyperbolic levels e_m. The latter are absent when $|B| \leqslant \frac{1}{2}$ (see [6]).

Now, we state our main result as follows:

Theorem 1. *Let $|B| \geqslant \frac{1}{2}$. Then, for $m \in Z_+ \cap [0, |B| - 1/2]$ we have that*

$$\mathcal{W}_{|B|,m}[\mathcal{H}] = E_{B,m}.$$

To show that $\mathcal{W}_{|B|,m}[\mathcal{H}] \subset E_{B,m}$ we need to compute the action of H_B on $\mathcal{W}_{|B|,m}[\varphi](\xi)$ for arbitrary function $\varphi \in \mathcal{H}$, $\xi = x + \mathrm{i}y \in \mathbf{H}^2$. A straightforward calculation gives

$$H_B(\mathcal{W}_{|B|,m}[\varphi](\xi)) = (|B| - m)(|B| - m - 1)\mathcal{W}_{|B|,m}[\varphi](\xi) + Q_{|B|,m}(\xi) \tag{4.2}$$

where

$$Q_{|B|,m}(\xi) = y^{|B|-m+1}\int_0^{+\infty} d\lambda(\varphi(\lambda))^*\lambda^{|B|-m}\, e^{-\frac{\lambda}{2}(y-ix)}\left(mL_m^{2(|B|-m)-1}(\lambda y)\right.$$
$$\left. + (2\lambda^{-1}(|B|-m)-y)\frac{\partial}{\partial y}\left(L_m^{2(|B|-m)-1}(\lambda y)\right) - \lambda^{-1}y\frac{\partial^2}{\partial y^2}\left(L_m^{2(|B|-m)-1}(\lambda y)\right)\right).$$

Using the relation (see [9], p 1037)

$$\frac{d}{dt}L_n^{\alpha}(t) = -L_{n-1}^{\alpha+1}(t)$$

then we can write the quantity $Q_{|B|,m}(\xi)$ as

$$Q_{|B|,m}(\xi) = y^{|B|-m+1}\int_0^{+\infty} d\lambda(\phi(\lambda))^*\lambda^{|B|-m}\, e^{-\frac{\lambda}{2}(y-ix)}\Phi_{|B|,m}(\lambda y)$$

where

$$\Phi_{|B|,m}(\lambda y) := mL_m^{2(|B|-m)-1}(\lambda y) - 2(|B|-m)L_{m-1}^{2(|B|-m)}(\lambda y)$$
$$+\lambda yL_{m-1}^{2(|B|-m)}(\lambda y) + \lambda yL_{m-2}^{2(|B|-m)+1}(\lambda y). \tag{4.3}$$

We may now apply the identity

$$L_n^{\alpha-1}(t) = L_n^{\alpha}(t) - L_{n-1}^{\alpha}(t)$$

to the first term of the above sum in (4.3) and apply the identity (see [9], p 1037)

$$tL_n^{\alpha+1}(t) = (n+\alpha+1)L_n^{\alpha}(t) - (n+1)L_{n+1}^{\alpha}(t)$$

to the last term in the same sum in (4.3) to get

$$\Phi_{|B|,m}(\lambda y) = mL_m^{2(|B|-m)}(\lambda y) - 2(|B|-1-\lambda y)L_{m-1}^{2(|B|-m)}(\lambda y)$$
$$+(m-1+2(|B|-m))L_{m-2}^{2(|B|-m)}(\lambda y).$$

Then, we make use of the functional relation ([9], p 1037)

$$(n+1)L_{n+1}^{\alpha}(t) - (2n+\alpha+1-t)L_n^{\alpha}(t) + (n+\alpha)L_{n-1}^{\alpha}(t) = 0$$

to find that $\Phi_{|B|,m}(\lambda y) = 0$ and therefore $Q_{|B|,m}(\xi) = 0$. In view of (4.2) it is clear that the function $\mathcal{W}_{|B|,m}[\varphi]$ is an eigenfunction of H_B corresponding to the eigenvalue $e_m = (|B|-m)(|B|-m-1)$.

Conversely, let $\psi \in E_{B,m}$. We consider the function

$$\varphi_\psi(\lambda) := c_{b,m}^{-\frac{1}{2}}\int_{\mathbf{H}^2} d\mu(\xi)(\psi(\xi))^*\langle\lambda \mid \xi\rangle_{B,m} \qquad \lambda \in \mathbf{R}_+^*.$$

We shall prove that φ_ψ is satisfies $\mathcal{W}_{|B|,m}[\varphi_\psi] = \psi$.

$$\mathcal{W}_{|B|,m}[\varphi_\psi](\xi) = \int_{\mathbf{R}_+^*}\frac{d\lambda}{\lambda}\left(\int_{\mathbf{H}^2} d\mu(\zeta)(\psi(\zeta))^*\langle\lambda \mid \zeta\rangle_{|B|,m}\right)^*\langle\lambda \mid \xi\rangle_{|B|,m}$$
$$= \int_{\mathbf{H}^2} d\mu(\zeta)\psi(\zeta)\int_{\mathbf{R}_+^*}\frac{d\lambda}{\lambda}\langle\lambda \mid \xi\rangle_{|B|,m}(\langle\lambda \mid \zeta\rangle_{|B|,m})^*$$
$$= \int_{\mathbf{H}^2} d\mu(\zeta)\psi(\zeta)\left(\frac{\Gamma(2|B|-m)}{m!}\right)^{-1}(\operatorname{Im}\zeta\,\operatorname{Im}\xi)^{|B|-m}$$
$$\times\int_0^{+\infty} d\lambda\exp\left(-\lambda\left[\frac{i(\operatorname{Re}\zeta-\operatorname{Re}\xi)}{2} + \frac{\operatorname{Im}\xi+\operatorname{Im}\zeta}{2}\right]\right)\lambda^{2(|B|-m)-1}$$
$$\times\ \lambda^{2(|B|-m)-1}L_m^{2(|B|-m)-1}(\lambda\operatorname{Im}\xi)L_m^{2(|B|-m)-1}(\lambda\operatorname{Im}\zeta).$$

Using the following identity ([9], p 845):

$$\int_0^{+\infty} \mathrm{e}^{-x(s+\frac{a_1+a_2}{2})} x^\alpha L_k^\alpha(a_1 x) L_k^\alpha(a_2 x)\,\mathrm{d}x = \frac{\Gamma(1+\alpha+k)}{b_0^{1+\alpha+k}} \frac{b_2^k}{k!} P_k^{(\alpha,0)}\left(\frac{b_1^2}{b_0 b_2}\right)$$

$$b_0 = s + \frac{a_1+a_2}{2} \qquad b_2 = s - \frac{a_1+a_2}{2}$$

$$b_1^2 = b_0 b_2 + 2a_1 a_2 \qquad \mathrm{Re}\,\alpha > -1 \qquad \mathrm{Re}\left(s + \frac{a_1+a_2}{2}\right) > 0$$

for $s = \frac{1}{2}\mathrm{i}(\mathrm{Re}\,\xi - \mathrm{Re}\,\zeta)$, $a_1 = \mathrm{Im}\,\xi$, $a_2 = \mathrm{Im}\,\zeta$, $\alpha = 2(|B| - m) - 1$ and $k = m$. $P_m^{(\alpha,\beta)}(.)$ is the Jacobi polynomial ([9], p 1035), which can also be expressed by means of the Gauss hypergeometric function ([9], p 1036):

$$P_m^{(\alpha,\beta)}(t) = \frac{\Gamma(m+1+\alpha)}{m!\Gamma(1+\alpha)}\, {}_2F_1\left(m+\alpha+\beta+1, -m, 1+\alpha, \frac{1-t}{2}\right).$$

After calculations, we obtain that

$$\mathcal{W}_{|B|,m}[\varphi_\psi](\xi) = \int_{\mathbf{H}^2} \mathrm{d}\mu(\zeta)\psi(\zeta)\, {}_2F_1\left(-2|B| - m, -m, 2|B| - 2m, \frac{4\mathrm{Im}\,\xi\ \mathrm{Im}\,\zeta}{|\xi - \zeta^*|^2}\right)$$
$$\times \frac{(-1)^m \Gamma(2|B| - m)}{m!\Gamma(2|B| - 2m)} \left(\frac{|\xi - \zeta^*|^2}{4\mathrm{Im}\,\xi\ \mathrm{Im}\,\zeta}\right)^{-|B|+m} \left(\frac{\zeta - \xi^*}{\xi - \zeta^*}\right)^{|B|}. \qquad (4.4)$$

But the function of variables (ξ, ζ) in the integral above coincides with the reproducing kernel of the eigenspace $E_{B,m}$ ([14], p 89). Therefore equation (4.4) gives that $\mathcal{W}_{|B|,m}[\varphi_\psi](\xi) = \psi(\xi)$. Finally, $\varphi_\psi \in \mathcal{H}$ since $\mathcal{W}_{|B|,m}$ is an isometry. We have then proved the inclusion $E_{B,m} \subset \mathcal{W}_{|B|,m}[\mathcal{H}]$.

5. Conclusion

We have been concerned with an infinite dimensional unitary irreducible representation of the affine group of the real line. This representation was realized on the Hilbert space of square integrable functions on the real positive half-line. By a group theoretical method, we construct a kind of generalized affine coherent state. Identifying the affine group with the complex upper half-plane, we establish that the image of the representation Hilbert space under the coherent state transforms, associated with the constructed coherent states, coincides with spaces of bound states of the Landau Hamiltonian on the hyperbolic plane. By this characterization, the coherent state method has provided us with a mathematical viewpoint of looking at eigenspaces of the Hamiltonian (Maass Laplacian). From another perspective, the established result indicates that we can increase our knowledge on properties of generalized affine coherent states by exploiting the accumulated mathematical findings at least since the early publications of Maass.

References

[1] Perelomov A 1986 *Generalized Coherent States and Their Applications* (New York: Springer)
[2] Aslaken E W and Klauder J R 1969 *J. Math. Phys* **10** 2267
[3] Daubechies I 1992 *Ten Lectures on Wavelets* (Philadelphia, PA: SIAM)
[4] Khalil I 1974 *Stud. Math.* **51** 140
[5] Kisil V V 1999 Relative convolutions: I. Properties and applications *Adv. Math.* **147** 35–73
[6] Comtet A 1987 On the Landau levels on the hyperbolic plane *Ann. Phys.* **173** 185–209
[7] Gel'fand I M and Naimark M A 1947 *Dokl. Akad. Nauk SSSR* **55** 570
[8] Duflo M and Moore C C 1976 *J. Funct. Anal.* **21** 208–43

[9] Gradshteyn I S and Ryzhik I M 1980 *Table of Integrals, Series and Products* (New York: Academic)
[10] Ali S T, Antoine J P and Gazeau J P 2000 *Coherent States, Wavelets and Their Generalization* (Berlin: Springer)
[11] Ikeda N and Matsumoto H 1999 *J. Funct. Anal.* **163** 63–263
[12] Oshima K 1989 *Prog. Theor. Phys.* **81** 286
[13] Elstrodt J 1973 *Math. Ann.* **203** 295
[14] Patterson S J 1975 *Comp. Math.* **31** 83